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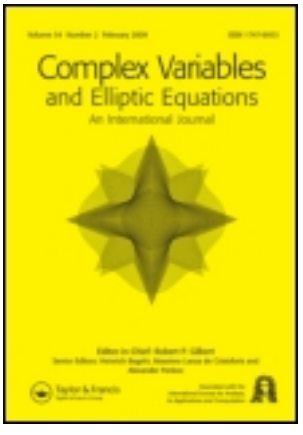

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Convexity of integral transforms and duality

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Convexity of integral transforms and duality

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For λ satisfying a certain admissibility criteria, sufficient conditions are obtained for the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\,\mathrm{d}t$$

to map normalized analytic functions f satisfying

$$\operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-\beta\right)>0$$

into the class of convex functions. Several interesting applications for different choices of λ are discussed. In particular, the smallest value $\beta<1$ is obtained that ensures a function f satisfying $\operatorname{Re}(f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z))>\beta$ is convex.

Keywords: duality; convolution; univalence; convex functions; integral transforms

AMS Subject Classifications: 30C45; 30C80

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disc $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ with the normalization $f(0)=0=f'(0)-1$, and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. A function $f\in\mathcal{A}$ is starlike if it maps $\mathbb{D}$ onto a starlike domain with respect to the origin, and f is convex if $f(\mathbb{D})$ is a convex domain. Analytically these are, respectively, equivalent to the conditions $\operatorname{Re}(zf'(z)/f(z))>0$ and $1+\operatorname{Re}(zf''(z)/f'(z))>0$ in $\mathbb{D}$. Denote by $\mathcal{S}^*$ and $\mathcal{CV}$ the classes of starlike and convex functions, respectively. A function $f\in\mathcal{A}$ is close-to-convex if there is a starlike function g and a real number α such that

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right)>0.$$

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The class of all such functions is denoted by $\mathcal{CC}$. For any two functions $f(z)=z+a_2z^2+\cdots$ and $g(z)=z+b_2z^2+\cdots$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f*g$ defined by

$$(f*g)(z) := z+\sum_{n=2}^{\infty} a_n b_n z^n.$$

For $f\in\mathcal{A}$, Fournier and Ruscheweyh [1] investigated starlikeness properties of the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\,\mathrm{d}t \tag{1}$$

over functions f in the class

$$\mathcal{P}(\beta) := \left\{f\in\mathcal{A} : \exists\phi\in\mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}(f'(z)-\beta)>0,\ z\in\mathbb{D}\right\}.$$

Here, λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)\mathrm{d}t = 1$. Ali and Singh [2] used the duality principle [3, 13] to find a sharp estimate of the parameter β that ensures $V_\lambda(f)$ is convex over $\mathcal{P}(\beta)$. In 2002, Choi et al. [4] investigated convexity property of the integral transform (1) over functions f in the class

$$\mathcal{P}_\alpha(\beta) := \left\{f\in\mathcal{A} : \exists\phi\in\mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left((1-\alpha)\frac{f(z)}{z}+\alpha f'(z)-\beta\right)>0,\ z\in\mathbb{D}\right\},$$

$\alpha\in\mathbb{R}$. The class $\mathcal{P}_\alpha(\beta)$ is closely related to the class $\mathcal{R}_\alpha(\beta)$ defined by

$$\mathcal{R}_\alpha(\beta) := \left\{f\in\mathcal{A} : \exists\phi\in\mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}(f'(z)+\alpha zf''(z)-\beta)>0,\ z\in\mathbb{D}\right\}.$$

It is evident that $f\in\mathcal{R}_\alpha(\beta)$ if and only if zf' belongs to $\mathcal{P}_\alpha(\beta)$.

Consider now the following class of functions that includes both classes $\mathcal{P}(\beta)$ and $\mathcal{P}_\alpha(\beta)$. For $\alpha\geq 0$, $\gamma\geq 0$ and $\beta<1$, define the class

$$\begin{aligned}\mathcal{W}_\beta(\alpha,\gamma) := \Big\{&f\in\mathcal{A} : \exists\phi\in\mathbb{R} \text{ with}\\ &\operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-\beta\right)>0,\ z\in\mathbb{D}\Big\}.\end{aligned} \tag{2}$$

Thus $\mathcal{P}(\beta)\equiv\mathcal{W}_\beta(1,0)$, $\mathcal{P}_\alpha(\beta)\equiv\mathcal{W}_\beta(\alpha,0)$, and $\mathcal{R}_\gamma(\beta)\equiv\mathcal{W}_\beta(1+2\gamma,\gamma)$. The class $\mathcal{W}_\beta(\alpha,\gamma)$ is closely related to the class $R(\alpha,\gamma,h)$ consisting of all solutions $f\in\mathcal{A}$ satisfying

$$f'(z)+\alpha zf''(z)+\gamma z^2f'''(z)\prec h(z),\quad z\in\mathbb{D},$$

with $h(z):=h_\beta(z)=(1+(1-2\beta)z)/(1-z)$. Here, $g(z)\prec h(z)$ indicates the function g is subordinate to h, or in other words, there is an analytic function w satisfying $w(0)=0$ and $|w(z)|<1$ such that $g(z)=h(w(z))$, $z\in\mathbb{D}$. When $\phi=0$ in (2), it is clear that $f\in R(\alpha,\gamma,h_\beta)$ if and only if zf' belongs to $\mathcal{W}_\beta(\alpha,\gamma)$. Every function $f\in R(\alpha,\gamma,h)$ for a suitably normalized convex function h has a double integral representation, which was recently investigated by Ali et al. [5].

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Interestingly, the general integral transform $V_\lambda(f)$ in (1) reduces to various well-known integral operators for specific choices of λ. For example,

$$\lambda(t) := (1+c)t^c, \quad c > -1$$

gives the Bernardi integral operator, while the choice

$$\lambda(t) := \frac{(a+1)^p}{\Gamma(p)} t^a \left(\log\frac{1}{t}\right)^{p-1}, \quad a > -1,\ p \geq 0$$

yields the Komatu operator [6]. Clearly, for $p=1$ the Komatu operator is the Bernardi operator.

For a certain choice of λ, the integral operator V_λ in (1) is the convolution between a function f and the Gaussian hypergeometric function $F(a,b;c;z) := {}_2F_1(a,b;c;z)$, which is related to the general Hohlov operator [7] given by

$$H_{a,b,c}(f) := zF(a,b;c;z) * f(z).$$

In the special case $a=1$, the operator reduces to the Carlson-Shaffer operator [8]. Here, ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function given by the series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad z \in \mathbb{D},$$

where the Pochhammer symbol is used to indicate $(a)_n = a(a+1)_{n-1}$, $(a)_0 = 1$, and where a, b, c are complex parameters with $c \neq 0, -1, -2, \ldots$.

In a recent paper, Ali et al. [9] investigated starlikeness properties of the integral transform (1) over the class $\mathcal{W}_\beta(\alpha,\gamma)$. This article investigates convexity of the integral transform V_λ over the class $\mathcal{W}_\beta(\alpha,\gamma)$ by applying the duality principle. Specifically, in Section 3, the best value $\beta < 1$ is determined that ensures V_λ maps $\mathcal{W}_\beta(\alpha,\gamma)$ into the class of convex functions $\mathcal{CV}$. Necessary and sufficient conditions are also derived that ensure $V_\lambda(f)$ is convex univalent. In Section 4, simpler sufficient conditions for $V_\lambda(f)$ to be convex are derived. These are used in Section 5 in the discussion of several interesting applications for specific choices of the admissible function λ. As a consequence, the smallest value $\beta < 1$ is obtained that ensures a function f satisfying $\mathrm{Re}(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)) > \beta$ is convex in the unit disc. The results obtained in this section extend and improve earlier works by several authors. The final section is devoted to extending the main convexity result to the generalized integral operator of the form $\rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$. The best value $\beta < 1$ is obtained that ensures the latter operator maps $\mathcal{W}_\beta(\alpha,\gamma)$ into the class $\mathcal{CV}$.

2. Preliminaries

We use the notations introduced in [9]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{3}$$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (3) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, choosing $\mu = 1$ gives $\nu = \gamma$.

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(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

In the sequel, whenever the particular case $\alpha = 1 + 2\gamma$ is considered, the values of μ and ν for $\gamma > 0$ will be taken as $\mu = 1$ and $\nu = \gamma$, respectively, while $\mu = 0$ and $\nu = 1 = \alpha$ in the case $\gamma = 0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n + 1} z^n, \tag{4}$$

and

$$\begin{aligned} \psi_{\mu,\nu}(z) = \phi_{\mu,\nu}^{-1}(z) &= 1 + \sum_{n=1}^{\infty} \frac{n + 1}{(n\nu + 1)(n\mu + 1)} z^n \\ &= \int_0^1 \int_0^1 \frac{\mathrm{d}s\,\mathrm{d}t}{(1 - t^{\nu} s^{\mu} z)^2}. \end{aligned} \tag{5}$$

Here, $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = 1/(1 - z)$. If $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$, and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{n\alpha + 1} z^n = \int_0^1 \frac{\mathrm{d}t}{(1 - t^{\alpha} z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^{\nu}$, $v = s^{\mu}$ result in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu - 1} v^{1/\mu - 1}}{(1 - uvz)^2} \mathrm{d}u\,\mathrm{d}v.$$

Thus, the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \dfrac{1}{\mu\nu} \displaystyle\int_0^1 \int_0^1 \frac{u^{1/\nu - 1} v^{1/\mu - 1}}{(1 - uvz)^2} \mathrm{d}u\,\mathrm{d}v, & \gamma > 0, \\ \displaystyle\int_0^1 \frac{\mathrm{d}t}{(1 - t^{\alpha} z)^2}, & \gamma = 0,\ \alpha \geq 0. \end{cases}$$

Now let q be the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} t^{1/\nu} q(t) = \begin{cases} \dfrac{1}{\mu\nu} t^{1/\nu - 1} \displaystyle\int_0^1 s^{1/\mu - 1} \frac{1 - st}{(1 + st)^3} \mathrm{d}s, & \gamma > 0, \\ \dfrac{1}{\alpha} t^{1/\alpha - 1} \dfrac{1 - t}{(1 + t)^3}, & \gamma = 0,\ \alpha > 0, \end{cases} \tag{6}$$

satisfying $q(0) = 0$. It is easily seen that the solution is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 s^{1/\mu - 1} w^{1/\nu - 1} \frac{1 - swt}{(1 + swt)^3} \mathrm{d}s\,\mathrm{d}w = \sum_{n=0}^{\infty} \frac{(n + 1)^2 (-1)^n t^n}{(1 + \mu n)(1 + \nu n)}. \tag{7}$$

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In particular,

$$q_\gamma(t) = \frac{1}{\gamma}\int_0^1 s^{1/\gamma-1}\frac{1}{(1+st)^2}\mathrm{d}s, \quad \gamma > 0,\ \alpha = 1+2\gamma,$$
$$q_\alpha(t) = \frac{1}{\alpha}t^{-1/\alpha}\int_0^t \tau^{1/\alpha-1}\frac{1-\tau}{(1+\tau)^3}\mathrm{d}\tau, \quad \gamma = 0,\ \alpha > 0. \tag{8}$$

3. Main results

Functions in the class $\mathcal{W}_\beta(\alpha,\gamma)$ generally are not convex. The following is the main result that gives conditions for convexity.

THEOREM 3.1 *Let $\mu \geq 0$, $\nu \geq 0$ satisfy* (3), *and let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)q(t)\mathrm{d}t, \tag{9}$$

where q is the solution of the initial-value problem (6). *Further let*

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}\mathrm{d}x, \quad \nu > 0, \tag{10}$$

$$\Pi_{\mu,\nu}(t) = \begin{cases} \displaystyle\int_t^1 \Lambda_\nu(x)x^{1/\nu-1-1/\mu}\,\mathrm{d}x, & \gamma > 0\ (\mu > 0,\ \nu > 0), \\ \Lambda_\alpha(t), & \gamma = 0\ (\mu = 0,\ \nu = \alpha > 0), \end{cases} \tag{11}$$

and assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Let V_λ be given by (1) *and*

$$h(z) = \frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon| = 1. \tag{12}$$

Then

$$\begin{cases} \mathrm{Re}\displaystyle\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)\mathrm{d}t > 0, & \gamma > 0, \\ \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)\mathrm{d}t > 0, & \gamma = 0, \end{cases} \tag{13}$$

if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{CV}$ for $f \in W_\beta(\alpha,\gamma)$. This conclusion does not hold for smaller values of β.

Proof Since the case $\gamma = 0$ ($\mu = 0$ and $\nu = \alpha$) corresponds to Lemma 3(ii) in [4, p. 121], it is sufficient to consider only the case $\gamma > 0$.

Let

$$H(z) = (1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z).$$

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Since $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$, then

$$H(z)=(1+\gamma-(\alpha-\gamma))\frac{f(z)}{z}+(\alpha-\gamma-\gamma)f'(z)+\gamma z f''(z)$$
$$=(1+\mu\nu-\mu-\nu)\frac{f(z)}{z}+(\mu+\nu-\mu\nu)f'(z)+\mu\nu z f''(z).$$

With $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, it follows from (4) that

$$H(z)=1+\sum_{n=1}^{\infty}a_{n+1}(n\nu+1)(n\mu+1)z^n=f'(z)*\phi_{\mu,\nu}(z), \tag{14}$$

and (5) yields

$$f'(z)=H(z)*\psi_{\mu,\nu}(z). \tag{15}$$

Let g be given by

$$g(z)=\frac{H(z)-\beta}{1-\beta}.$$

Since $\operatorname{Re} e^{i\phi}g(z)>0$, the duality principle allows us to assume that

$$g(z)=\frac{1+xz}{1+yz},\quad |x|=1,\quad |y|=1. \tag{16}$$

Now, (15) implies that $f'(z)=[(1-\beta)g(z)+\beta]*\psi_{\mu,\nu}(z)$, and (16) readily gives

$$\frac{f(z)}{z}=\frac{1}{z}\int_0^z\left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w*\psi(z), \tag{17}$$

where for convenience, we write $\psi:=\psi_{\mu,\nu}$.

If $f\in\mathcal{W}_\beta(\alpha,\gamma)$, a well-known result in [3, p. 94] states that

$$F\in S^*\Longleftrightarrow\frac{1}{z}(F*h)(z)\neq 0,\quad z\in\mathbb{D},$$

where h is given by (12). Now $F\in\mathcal{CV}$ if and only if $zF'\in S^*$, and thus

$$0\neq\frac{1}{z}(zF'(z)*h(z))$$
$$=\frac{1}{z}(F(z)*zh'(z))=\frac{1}{z}\left[\int_0^1\lambda(t)\frac{f(tz)}{t}\mathrm{d}t*zh'(z)\right]$$
$$=\int_0^1\frac{\lambda(t)}{1-tz}\mathrm{d}t*\frac{f(z)}{z}*h'(z).$$

From (17), it follows that

$$0\neq\int_0^1\frac{\lambda(t)}{1-tz}\mathrm{d}t*\left[\frac{1}{z}\int_0^z\left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w*\psi(z)\right]*h'(z)$$
$$=\int_0^1\frac{\lambda(t)}{1-tz}\mathrm{d}t*h'(z)*\left[\frac{1}{z}\int_0^z\left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)\mathrm{d}w\right]*\psi(z)$$

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$$= \int_0^1 \lambda(t)h'(tz)\mathrm{d}t * (1-\beta)\left[\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}\mathrm{d}w + \frac{\beta}{1-\beta}\right] * \psi(z)$$
$$= (1-\beta)\left[\int_0^1 \lambda(t)h'(tz)\mathrm{d}t + \frac{\beta}{1-\beta}\right] * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}\mathrm{d}w * \psi(z).$$

It is known [3, p. 23] that the dual set of functions g given by (16) consists of analytic functions p satisfying $p(0)=1$ and $\mathrm{Re}\ p(z) > 1/2$ in $\mathbb{D}$. Hence

$$0 \neq (1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{1-\beta}\right] * \frac{1+xz}{1+yz} * \psi(z)$$

$$\Longleftrightarrow \mathrm{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{1-\beta}\right] * \psi(z) > \frac{1}{2}$$
$$\Longleftrightarrow \mathrm{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)\mathrm{d}w\right)\mathrm{d}t + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right] * \psi(z) > 0.$$

Using (9), the latter condition is equivalent to

$$\mathrm{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)\mathrm{d}w - q(t)\right)\mathrm{d}t\right] * \psi(z) > 0.$$

From (5), the above inequality is equivalent to

$$0 < \mathrm{Re}\int_0^1 \lambda(t)\left(\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * h'(tz) - q(t)\right)\mathrm{d}t$$
$$= \mathrm{Re}\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{\mathrm{d}\eta\,\mathrm{d}\zeta}{1-z\eta^{\nu}\zeta^{\mu}} * h'(tz) - q(t)\right)\mathrm{d}t$$
$$= \mathrm{Re}\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 h'(tz\eta^{\nu}\zeta^{\mu})\mathrm{d}\eta\,\mathrm{d}\zeta - q(t)\right)\mathrm{d}t,$$

which reduces to

$$\mathrm{Re}\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{1}{\mu\nu}h'(tzuv)u^{1/\nu-1}v^{1/\mu-1}\mathrm{d}v\,\mathrm{d}u - q(t)\right]\mathrm{d}t > 0.$$

A change of variable $w = tu$ leads to

$$\mathrm{Re}\int_0^1 \frac{\lambda(t)}{t^{1/\nu}}\left[\int_0^t\int_0^1 h'(wzv)w^{1/\nu-1}v^{1/\mu-1}\mathrm{d}v\,\mathrm{d}w - \mu\nu t^{1/\nu}q(t)\right]\mathrm{d}t > 0.$$

Integrating by parts with respect to t and using (6) gives the equivalent form

$$\mathrm{Re}\int_0^1 \Lambda_{\nu}(t)\left[\int_0^1 h'(tzv)t^{1/\nu-1}v^{1/\mu-1}\mathrm{d}v - t^{1/\nu-1}\int_0^1 s^{1/\mu-1}\frac{1-st}{(1+st)^3}\mathrm{d}s\right]\mathrm{d}t > 0.$$

Making the variable change $w = vt$ and $\eta = st$ reduces the above inequality to

$$\mathrm{Re}\int_0^1 \Lambda_{\nu}(t)t^{1/\nu-1/\mu-1}\left[\int_0^t h'(wz)w^{1/\mu-1}\mathrm{d}w - \int_0^t \eta^{1/\mu-1}\frac{1-\eta}{(1+\eta)^3}\mathrm{d}\eta\right]\mathrm{d}t > 0,$$

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which after integrating by parts with respect to t yields

$$\operatorname{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)\mathrm{d}t>0.$$

Thus $F\in\mathcal{CV}$ if and only if condition (13) holds.

To verify sharpness, let β_0 satisfy

$$\frac{\beta_0-1/2}{1-\beta_0}=-\int_0^1\lambda(t)q(t)\mathrm{d}t.$$

Assume that $\beta<\beta_0$ and let $f\in\mathcal{W}_\beta(\alpha,\gamma)$ be the solution of the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)=\beta+(1-\beta)\frac{1+z}{1-z}.$$

From (14), it follows that

$$f(z)=z+\sum_{n=1}^{\infty}\frac{2(1-\beta)}{(n\nu+1)(n\mu+1)}z^{n+1},$$

and

$$G(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}\mathrm{d}t=z+\sum_{n=1}^{\infty}\frac{2(1-\beta)}{(n\nu+1)(n\mu+1)}\left(\int_0^1\lambda(t)t^n\,\mathrm{d}t\right)z^{n+1}.$$

Thus

$$G(z)=V_\lambda(f)(z)=z+\sum_{n=1}^{\infty}\frac{2(1-\beta)\tau_n}{(n\nu+1)(n\mu+1)}z^{n+1},$$

where $\tau_n=\int_0^1\lambda(t)t^n\mathrm{d}t$. Now (7) implies that

$$\frac{\beta_0-1/2}{1-\beta_0}=-\int_0^1\lambda(t)q(t)\mathrm{d}t=-\sum_{n=1}^{\infty}\frac{(n+1)^2(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}.$$

This means that

$$(zG')'\Big|_{z=-1}=1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)^2(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}=1-\frac{1-\beta}{1-\beta_0}<0.$$

Hence $(zG')'(z)=0$ for some $z\in\mathbb{D}$, and so zG' is not even locally univalent in $\mathbb{D}$. Therefore the value of β in (9) is sharp. ■

Remark 1 Theorem 3.1 yields several known results.

(1) When $\gamma=0$, then $\mu=0$, $\nu=\alpha$, and in this particular instance, Theorem 3.1 gives Lemma 3(ii) in Choi et al. [4, p. 121]. There the range of α lies in [1/2, 1], whereas the range of α in Theorem 3.1 for this particular case is $\alpha>0$.

(2) The special case $\alpha=1$ above yields a result of Ali and Singh [2, Theorem 1(ii), p. 301].

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If $\alpha = 1 + 2\gamma$, then $\mu = 1$ and $\nu = \gamma$ for $\gamma > 0$, while $\mu = 0$ and $\nu = \alpha = 1$ for $\gamma = 0$. In this instance, Theorem 3.1 gives the following result.

COROLLARY 3.2 *Let $f \in \mathcal{W}_\beta(1+2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, $\gamma \geq 0$, and let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q_\gamma(t) \mathrm{d}t,$$

where q_γ is given by (8). *Further, let Λ_γ be defined by* (10),

$$\Pi_\gamma(t) = \begin{cases} \displaystyle\int_t^1 \Lambda_\gamma(s) s^{1/\gamma - 2} \mathrm{d}s, & \gamma > 0\ , \\ \displaystyle\int_t^1 \frac{\lambda(s)}{s} \mathrm{d}s, & \gamma = 0\ , \end{cases} \tag{18}$$

and h be given by (12). *Then*

$$\mathrm{Re} \int_0^1 \Pi_\gamma(t) \left(h'(tz) - \frac{1 - t}{(1 + t)^3} \right) \mathrm{d}t > 0,$$

if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{CV}$. The conclusion does not hold for smaller values of β.

4. Convexity criteria of integral transforms

The conditions stipulated in Theorem 3.1 can be cumbersome to use. A simpler sufficient condition for convexity of the integral operator (1) is now given in the following theorem. The following lemma of Fournier and Ruscheweyh [1, Theorem 1, p. 530] is required.

LEMMA 4.1 *Let Λ be integrable on $[0, 1]$, and positive on $(0, 1)$. If $\Lambda(t)/(1 - t^2)$ is decreasing on $(0, 1)$, then $L_\Lambda(\mathcal{CC}) = 0$, where*

$$L_\Lambda(f) = \inf_{z \in D} \int_0^1 \Lambda(t) \left(\mathrm{Re} \frac{f(tz)}{tz} - \frac{1}{(1 + t)^2} \right) \mathrm{d}t \quad (f \in \mathcal{S})$$

and

$$L_\Lambda(\mathcal{CC}) = \inf_{f \in \mathcal{CC}} L_\Lambda(f).$$

THEOREM 4.2 *Let $\Pi_{\mu,\nu}$ and Λ_ν be given as in Theorem* 3.1. *Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on $[0, 1]$, and positive on $(0, 1)$. Assume further that $\mu \geq 1$ and*

$$\frac{\Lambda_\nu(t) t^{1/\nu - 1/\mu} + (1 - 1/\mu) \Pi_{\mu,\nu}(t)}{1 - t^2} \quad \text{is decreasing on } (0, 1). \tag{19}$$

If β satisfies (9), *and $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then $V_\lambda(f) \in \mathcal{CV}$.*

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Proof Integrating by parts with respect to t yields

$$\begin{aligned}
&\mathrm{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)\mathrm{d}t\\
&=\mathrm{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\frac{\partial}{\partial t}\left(\frac{h(tz)}{z}-\frac{t}{(1+t)^2}\right)\mathrm{d}t\\
&=\mathrm{Re}\int_0^1 t^{1/\mu-1}\left(\Lambda_\nu(t)t^{1/\nu-1/\mu}+\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)\mathrm{d}t.
\end{aligned}$$

The function $t^{1/\mu-1}$ is decreasing on $(0,1)$ when $\mu\geq 1$. Thus, the condition (19) along with Lemma 4.1 yields

$$\mathrm{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)\mathrm{d}t>0.$$

The desired conclusion now follows from Theorem 3.1. ■

Let us scrutinize Theorem 4.2 for helpful conditions to ensure convexity of $V_\lambda(f)$. Now for $\gamma>0$,

$$\Pi_{\mu,\nu}(t)=\int_t^1 \Lambda_\nu(y)\,y^{1/\nu-1-1/\mu}\,\mathrm{d}y \quad\text{and}\quad \Lambda_\nu(t)=\int_t^1 \frac{\lambda(x)}{x^{1/\nu}}\mathrm{d}x.$$

To apply Theorem 4.2, it is sufficient to show that the function

$$k(t)=\frac{\Lambda_\nu(t)t^{1/\nu-1/\mu}+(1-1/\mu)\Pi_{\mu,\nu}(t)}{1-t^2}:=\frac{p(t)}{1-t^2}$$

is decreasing in the interval $(0,1)$. Note that $k(t)>0$ and decreasing in the interval $(0,1)$ provided

$$q(t):=p(t)+\frac{1-t^2}{2}t^{-1}p'(t)\leq 0.$$

Since $q(1)=0$, this will certainly hold if q is increasing in $(0,1)$. Now

$$q'(t)=\frac{1-t^2}{2}t^{-2}[tp''(t)-p'(t)],$$

and

$$\begin{aligned}
tp''(t)-p'(t)=&-\lambda(t)t^{-1/\mu}\left(\frac{1}{\nu}-\frac{1}{\mu}-2+\frac{t\lambda'(t)}{\lambda(t)}\right)\\
&+\left(\frac{1}{\nu}-\frac{1}{\mu}-2\right)\left(\frac{1}{\nu}-1\right)t^{1/\nu-1/\mu-1}\Lambda_\nu(t).
\end{aligned}$$

Thus, $tp''(t)-p'(t)$ is non-negative if

$$\frac{1}{\nu}-\frac{1}{\mu}-2+\frac{t\lambda'(t)}{\lambda(t)}\leq 0 \quad\text{and}\quad \left(\frac{1}{\nu}-\frac{1}{\mu}-2\right)\left(\frac{1}{\nu}-1\right)\geq 0. \tag{20}$$

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For $\mu \geq 1$, condition (3) implies $\nu \geq \mu \geq 1$. Thus, condition (20) is equivalent to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1.$$

These observations result in the following theorem.

THEOREM 4.3 *Let λ be a non-negative real-valued integrable function on $[0,1]$. Assume that Λ_ν and $\Pi_{\mu,\nu}$ given by (10) and (11) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 3.1, if λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1, \tag{21}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

Remark 1 The condition $\mu \geq 1$ is equivalent to $0 < \gamma \leq \alpha \leq 2\gamma + 1$.

Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 4.3 yields the following result.

COROLLARY 4.4 *Let λ be a non-negative real-valued integrable function on $[0,1]$. Let $f \in \mathcal{W}_\beta(1+2\gamma, \gamma) = \mathcal{R}_\beta(\gamma)$, $\gamma \in [1, \infty)$, and let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q_\gamma(t) \mathrm{d}t,$$

where q_γ is given by (8). Assume further that $\Pi_{1,\gamma}$ and Λ_γ are integrable on $[0,1]$ and positive on $(0,1)$. If λ satisfies

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 3 - \frac{1}{\gamma},$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

In the case $\gamma = 0$ and $\alpha \geq 1$ ($\mu = 0, \nu = \alpha$), an easier sufficient condition for convexity of the integral operator (1) is obtained in the following theorem.

THEOREM 4.5 *Let λ be a non-negative real-valued integrable function on $[0,1]$. Assume that Λ_α and $\Pi_{0,\alpha}$ given by (10) and (11) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 3.1, if $\lambda(1) = 0$ and λ satisfies*

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) > 0, \quad \alpha \geq 1, \tag{22}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

Proof From Theorem 3.1, it suffices to show that

$$\mathrm{Re} \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) \mathrm{d}t > 0, \quad \gamma = 0.$$

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Integrating by parts with respect to t yields

$$\operatorname{Re}\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)\mathrm{d}t$$
$$=\operatorname{Re}\int_0^1 t^{1/\alpha-1}\left(t^{1-1/\alpha}\lambda(t)+\left(1-\frac{1}{\alpha}\right)\Lambda_\alpha(t)\right)\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)\mathrm{d}t.$$

The function $t^{1/\alpha-1}$ is decreasing on $(0,1)$ when $\alpha\geq 1$. Thus, the condition

$$\frac{t^{1-1/\alpha}\lambda(t)+\left(1-\frac{1}{\alpha}\right)\Lambda_\alpha(t)}{1-t^2} \qquad \text{is decreasing on } (0,1)$$

along with Lemma 4.1 will yield

$$\operatorname{Re}\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)\mathrm{d}t>0.$$

Let $p(t)=k(t)/(1-t^2)$, where $k(t)=t^{1-1/\alpha}\lambda(t)+(1-1/\alpha)\Lambda_\alpha(t)$. Taking the logarithmic derivative of p and using the fact that $p(t)>0$ for $\alpha\geq 1$, the condition $p'(t)\leq 0$ in $(0,1)$ is equivalent to the inequality

$$q(t)=k(t)+t^{-1}k'(t)\frac{1-t^2}{2}\leq 0.$$

Clearly $q(1)=0$ and if q is increasing in $(0,1)$, then p will be decreasing in $(0,1)$. Direct computations show that $q'(t)\geq 0$ provided $(t^{-1}k'(t))'\geq 0$. Since

$$(t^{-1}k'(t))'=t^{-1/\alpha-1}\left(t\lambda''(t)-\frac{1}{\alpha}\lambda'(t)\right),$$

the desired result follows from (22). ■

5. Applications to several integral transforms

In this section, various well-known integral operators are considered, and conditions for convexity for $f\in\mathcal{W}_\beta(\alpha,\gamma)$ under these integral operators are obtained. First let λ be defined by

$$\lambda(t)=(1+c)t^c,\quad c>-1.$$

Then the integral transform

$$F_c(z)=V_\lambda(f)(z)=(1+c)\int_0^1 t^{c-1}f(tz)\mathrm{d}t,\quad c>-1 \tag{23}$$

is the Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (23) with $c=0$ and $c=1$, respectively. For this special case of λ, the following result holds.

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THEOREM 5.1 *Let $c > -1$, $0 < \gamma \le \alpha \le 1 + 2\gamma$, and $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -(c+1)\int_0^1 t^c q(t)\mathrm{d}t,$$

where q is given by (7). *If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)\mathrm{d}t$$

belongs to $\mathcal{CV}$ provided

$$c \le 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad \nu \ge \mu \ge 1.$$

The value of β is sharp.

Proof With $\lambda(t) = (1+c)t^c$, then $t\lambda'(t)/\lambda(t) = c$, and the result readily follows from Theorem 4.3. ■

When $\alpha = 1 + 2\gamma$, $\gamma > 0$, and $\mu = 1$, Theorem 23 yields the following result.

COROLLARY 5.2 *Let $-1 < c \le 3 - 1/\gamma$, $\gamma \in [1, \infty)$, and $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -(c+1)\int_0^1 t^c q_\gamma(t)\mathrm{d}t,$$

where q_γ is given by (8). *If $f \in \mathcal{W}_\beta(1 + 2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, then the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)\mathrm{d}t$$

belongs to $\mathcal{CV}$. The value of β is sharp.

The case $c = 0$ in Theorem 5.1 yields the following interesting result, which we state as a theorem.

THEOREM 5.3 *Let $0 < \gamma \le \alpha \le 1 + 2\gamma$. If $F \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\big(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)\big) > \beta$$

in $\mathbb{D}$, and $\beta < 1$ satisfies

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 q(t)\mathrm{d}t,$$

where q is given by (7), *then F is convex. The value of β is sharp.*

Proof It is evident that the function $f = zF'$ belongs to the class

$$\mathcal{W}_{\beta,0}(\alpha, \gamma) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left((1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) \right) > \beta,\ z \in \mathbb{D} \right\}.$$

Thus

$$F(z) = \int_0^1 \frac{f(tz)}{t}\mathrm{d}t.$$

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The conditions on α and γ imply that $1 \le \mu \le \nu$. Thus the result now follows from Theorem 5.1 with $c = 0$. It is also evident from the proof of sharpness in Theorem 3.1 that the extremal function in $\mathcal{W}_\beta(\alpha, \gamma)$ indeed also belongs to the class $\mathcal{W}_{\beta,0}(\alpha, \gamma)$. ■

Example 5.4 If $\gamma = 1$, $\alpha = 3$, then $\mu = 1 = \nu$. In this case, (7) yields $\beta = (1 - 2\ln 2)/2(1 - \ln 2) = -0.629445$. Thus

$$\mathrm{Re}\big(f'(z) + 3zf''(z) + z^2 f'''(z)\big) > \beta \Rightarrow f \in \mathcal{CV}.$$

THEOREM 5.5 *Let $b > -1$, $a > -1$ and $0 < \gamma \le \alpha \le 2\gamma + 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q(t) \mathrm{d}t,$$

where q is given by (7) *and*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\dfrac{t^a(1 - t^{b-a})}{b-a}, & b \ne a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases} \tag{24}$$

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then

$$\mathcal{G}_f(a, b; z) = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} \displaystyle\int_0^1 t^{a-1}(1 - t^{b-a}) f(tz) \mathrm{d}t, & b \ne a, \\ (a+1)^2 \displaystyle\int_0^1 t^{a-1} \log(1/t) f(tz) \mathrm{d}t, & b = a, \end{cases} \tag{25}$$

belongs to $\mathcal{CV}$ provided

$$a \le 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \ge \mu \ge 1. \tag{26}$$

The value of β is sharp.

Proof It is easily seen that $\int_0^1 \lambda(t) \mathrm{d}t = 1$. There are two cases to consider. When $b \ne a$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(b-a)t^{b-a}}{1 - t^{b-a}}.$$

The function λ satisfies (21) if

$$a - \frac{(b-a)t^{b-a}}{1 - t^{b-a}} \le 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \ge \mu \ge 1. \tag{27}$$

Since $t \in (0, 1)$, the condition $b > a$ implies $(b-a)t^{b-a}/(1 - t^{b-a}) > 0$, and so inequality (27) holds true whenever a satisfies (26). When $b < a$, then $(a-b)/(t^{a-b} - 1) < b - a$, and hence $a - (b-a)t^{b-a}/(1 - t^{b-a}) < b < a$, and thus (27) holds if a satisfies (26).

For the case $b = a$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{1}{\log(1/t)}.$$

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Since $t < 1$ implies $1/\log(1/t) \geq 0$, condition (21) is satisfied provided a satisfies (26). This completes the proof. ■

The simpler condition (21) can also be applied to the choice

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a \left(\log\left(\frac{1}{t}\right)\right)^{p-1}, \quad a > -1,\ p \geq 0.$$

The integral transform V_λ in this case takes the form

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)\mathrm{d}t, \quad a > -1,\ p \geq 0.$$

This is the Komatu operator, which reduces to the Bernardi integral operator when $p = 1$. For this λ, the following result holds.

THEOREM 5.6 *Let $a > p - 2 \geq -1$ and $0 < \gamma \leq \alpha \leq 2\gamma + 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \left(\log\left(\frac{1}{t}\right)\right)^{p-1} q(t)\mathrm{d}t,$$

where q is given by (7). *For $f \in \mathcal{W}_\beta(\alpha, \gamma)$, the function*

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)\mathrm{d}t \tag{28}$$

belongs to $\mathcal{CV}$ provided

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \tag{29}$$

The value of β is sharp.

Proof Brief computations show that

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(p-1)}{\log(1/t)}.$$

Since $\log(1/t) > 0$ for $t \in (0, 1)$, and $p \geq 1$, condition (21) is satisfied whenever a satisfies (29). ■

We next apply Theorem 4.5 to the case $\lambda(1) = 0$ as shown by the following two theorems.

THEOREM 5.7 *Suppose $a > -1$, $b > -1$, $\alpha \geq 1$ are related by*

(1) $-1 < a \leq 0$ *and* $a = b$, *or*
(2) $-1 < a \leq 0$ *and* $-1 < a < b \leq 1 + 1/\alpha$.

Let $\beta < 1$ satisfy

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t) q(t)\mathrm{d}t,$$

where q and λ are given by (8) *and* (24), *respectively. If $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\beta(\alpha)$, then $\mathcal{G}_f(a, b; z)$ defined by* (25) *belongs to $\mathcal{CV}$. The value of β is sharp.*

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Proof To apply Theorem 4.5, it suffices to verify inequality (22) for λ defined by (24). It is seen that

$$\lambda'(t) = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} t^{a-1}(a - bt^{b-a}), & b > a, \\ (a+1)^2\left(-1 + a\log\left(\dfrac{1}{t}\right)\right)t^{a-1}, & b = a, \end{cases}$$

and

$$t\lambda''(t) = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} t^{a-1}(a(a-1) - b(b-1)t^{b-a}), & b > a, \\ (a+1)^2\left(1 - 2a + a(a-1)\log\left(\dfrac{1}{t}\right)\right)t^{a-1}, & b = a. \end{cases}$$

Case i Let $b = a > -1$. Substituting the expression for λ' and $t\lambda''$ in (22) yields the equivalent condition

$$a\log\left(\frac{1}{t}\right)\left(a - \frac{1}{\alpha} - 1\right) + \left(-2a + \frac{1}{\alpha} + 1\right) \geq 0.$$

This clearly holds for $t \in (0,1)$ whenever $-1 < a \leq \min\{0, 1+1/\alpha, (1+1/\alpha)/2\} = 0$.

Case ii Let $b > a > -1$ with $a \in (-1, 0]$ and $-1 < b \leq 1 + 1/\alpha$. In this case, condition (22) is equivalent to $\psi_t(a) \geq \psi_t(b)$, where

$$\psi_t(a) = a(a-1)t^a - \frac{1}{\alpha}at^a.$$

For a fixed t,

$$\psi_t'(a) = t^a\left(2a - 1 - \frac{1}{\alpha} - a^2\log\left(\frac{1}{t}\right) + a\left(1 + \frac{1}{\alpha}\right)\log\left(\frac{1}{t}\right)\right),$$

that is, $\psi_t'(a) \leq 0$ for $a \in (-1, 0)$. Thus, $\psi_t(a)$ is a decreasing function of a for each fixed $t \in (0,1)$. In particular, for $b > a$ with $b \in (-1, 0)$ and $a \in (-1, 0)$, inequality (22) holds. When $b > a$ with $0 < b \leq 1 + 1/\alpha$, then $\psi_t(a) \geq \psi_t(0) = 0$ for each fixed $t \in (0,1)$. For $0 < b \leq 1 + 1/\alpha$,

$$\psi_t(b) = bt^b\left(b - 1 - \frac{1}{\alpha}\right) \leq 0.$$

It follows then that $\psi_t(a) \geq 0 \geq \psi_t(b)$ holds for each fixed $t \in (0,1)$. Thus, inequality (22) holds for $b > a > -1$ with $a \in (-1, 0]$ and $0 < b \leq 1 + 1/\alpha$. ■

Remark 1 The conditions $b > -1$ and $a > -1$ in Theorem 5.7 yield several improvements of known results.

(1) Taking $\gamma = 0$ and $\alpha \geq 1$ in Theorem 5.7 leads to a result extending Theorem 3.4 obtained in [10, p. 12] for the case $\alpha \in [1/2, 1]$. When $\alpha = 1$,

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the range of b there lies in $(-1, 0]$, whereas the range of b in Theorem 5.7 lies in the larger interval $(-1, 2]$.

(2) $\alpha = 1$ above leads to improvement of a result obtained by Ponnusamy and Rønning [11, Corollary 3.2]. There the parameters $b > a > -1$ must satisfy a fairly complicated equation to deduce $\mathcal{G}_f(a, b; z)$ is starlike of order $1/2$. In our present situation, the conditions on the parameters a and b are simpler to infer convexity.

(3) For $\alpha = 1$, $\gamma = 0$, $a = -\eta$ and $b = -\eta + 2$, Theorem 5.7 reduces to Corollary 1 [2, p. 302] and Corollary 1 in [12, pp. 915–916, $(\gamma = 0)$].

Now let Φ be defined by $\Phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n (1-t)^n$, $b_n \geq 0$ for $n \geq 1$, and

$$\lambda(t) = K t^{b-1}(1-t)^{c-a-b}\Phi(1-t), \tag{30}$$

where K is a constant chosen such that $\int_0^1 \lambda(t)\mathrm{d}t = 1$.

THEOREM 5.8 *Let $a, b, c > 0$ and $\alpha \geq 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)q(t)\mathrm{d}t,$$

where q is given by (8)*, and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(\alpha, 0)$, then the function*

$$V_\lambda(f)(z) = K\int_0^1 t^{b-2}(1-t)^{c-a-b}\Phi(1-t)f(tz)\mathrm{d}t$$

belongs to $\mathcal{CV}$ provided

$$c \geq a + b + 1 \quad \textit{and} \quad 0 < b \leq 1.$$

The value of β is sharp.

Proof As in the earlier proof, it suffices to verify inequality (22). Consider λ given by (30). Direct computations show that

$$\lambda'(t) = K t^{b-2}(1-t)^{c-a-b-1}\Big(\big((b-1)(1-t) - (c-a-b)t\big)\Phi(1-t) - t(1-t)\Phi'(1-t)\Big),$$

and

$$\begin{aligned} t\lambda''(t) = K t^{b-2}(1-t)^{c-a-b-2}\Big(&\big((b-1)(b-2)(1-t)^2 \\ &- 2(b-1)(c-a-b)t(1-t) + (c-a-b)(c-a-b-1)t^2\big)\Phi(1-t) \\ &+ \big(2(c-a-b)t - 2(b-1)(1-t)\big)t(1-t)\Phi'(1-t) + t^2(1-t)^2\Phi''(1-t)\Big). \end{aligned}$$

Thus, (22) is satisfied provided

$$\psi(t) = \Phi(1-t)X(t) + t(1-t)\Phi'(1-t)Y(t) + t^2(1-t)^2\Phi''(1-t) \geq 0,$$

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where

$$X(t) = (1-t)^2(b-1)\left(-\frac{1}{\alpha}+b-2\right) - (c-a-b)t(1-t)\left(-\frac{1}{\alpha}+2b-2\right) + (c-a-b)(c-a-b-1)t^2,$$

$$Y(t) = 2(c-a-b)t + (1-t)\left(\frac{1}{\alpha}-2b+2\right).$$

Since $\Phi(1-t) = 1 + \Sigma_{n=1}^{\infty} b_n(1-t)^n$, $b_n \geq 0$ for $n \geq 1$, the functions $\Phi(1-t)$, $\Phi'(1-t)$ and $\Phi''(1-t)$ are non-negative for $t \in (0,1)$. Therefore, it suffices to show

$$X(t) \geq 0 \quad \text{and} \quad Y(t) \geq 0,$$

and these evidently hold provided $c \geq a+b+1$ and $0 < b \leq \min\{1, 2+1/\alpha, (2+1/\alpha)/2\} = 1$. ■

Remark 2 For $\gamma = 0$ and $\alpha \geq 1$, Theorem 5.8 extends Theorem 3.1 in [10, p. 9, ($\mu = 0$)] for $\alpha \in [1/2, 1]$. When $\alpha = 1$, the range of b obtained in [10] lies in the interval $(0, 1/2]$, whereas the range of b obtained in Theorem 5.8 for this particular case lies in $(0, 1]$.

Remark 3 As shown in [10], choosing

$$\Phi(1-t) = F(c-a, 1-a, c-a-b+1; 1-t)$$

gives

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.$$

In this case, $V_\lambda(f)$ reduces to the Hohlov operator given by

$$V_\lambda(f)(z) = H_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) = K\int_0^1 t^{b-2}(1-t)^{c-a-b}F(c-a, 1-a, c-a-b+1; 1-t)f(tz)\mathrm{d}t,$$

where $a > 0$, $b > 0$ and $c-a-b+1 > 0$. In particular, for $a = 1$, $b = a$ and $c = a+b$, Theorem 5.8 yields Corollary 2 in [2, p. 302] and Corollary 2 in [12, p. 916, ($\gamma = 0$)]. In the case $\gamma = 0$ and $\alpha \geq 1$, Theorem 5.8 extends Theorem 1 in [4, p. 122] and Theorem 3.2 in [10, p. 11]. When $\alpha = 1$, the range of b obtained in [10] lies in the interval $(0, 1/2]$, whereas the range of b in Theorem 5.8 lies in $(0, 1]$. This result improves Theorem 1 as well obtained by Choi et al. [4] for the particular case $\alpha = 1$.

Choosing now [10, Theorem 3.3, p. 12]

$$\Phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1}$$

in Theorem 5.8 yields the following interesting result, which we state as a theorem.

THEOREM 5.9 *Let* $-1 < a \leq 0$, $\alpha \geq 1$ *and* $p \geq 2$. *Let* $\beta < 1$ *satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)}\int_0^1 t^a\left(\log\frac{1}{t}\right)^{p-1} q(t)\mathrm{d}t,$$

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where q is given by (8). *If* $f \in \mathcal{W}_\beta(\alpha, 0)$, *then the function* $\Phi_p(a; z) * f(z)$ *defined by* (28) *belongs to* $\mathcal{CV}$. *The value of* β *is sharp.*

Proof Choose

$$\Phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1},$$

$c - a - b = p - 1$ and $b = a + 1$ so that λ defined by Equation (30) takes the form

$$\lambda(t) = Kt^a(1-t)^{p-1}\Phi(1-t), \quad K = \frac{(1+a)^p}{\Gamma(p)}.$$

The desired result now follows from Theorem 5.8. ■

Remark 4 For the particular case $\alpha = 1$, this result improves Theorem 3.3 by Balasubramanian et al. [10], where the range of a obtained there has been improved from $(-1, -1/2]$ to $(-1, 0]$.

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6. Starlikeness and convexity of a generalized operator

In [2], Ali and Singh generalized the operator (1) to the case

$$\mathcal{V}_\lambda(f)(z) := \rho z + (1-\rho)V_\lambda(f) = z\int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}\mathrm{d}t * f(z), \quad \rho < 1. \tag{31}$$

In this final section, Theorem 3.1 is generalized to obtain conditions on λ such that $\mathcal{V}_\lambda(f) \in \mathcal{S}^*$ or $\mathcal{CV}$ for $f \in \mathcal{W}_\beta(\alpha, \gamma)$. The proofs are similar to the proof of Theorem 3.1, and are therefore omitted.

THEOREM 6.1 *Let* $\Pi_{\mu,\nu}$ *and* Λ_ν *be given as in Theorem* 3.1. *Assume that both* $\Pi_{\mu,\nu}$ *and* Λ_ν *are integrable on* $[0, 1]$, *and positive on* $(0, 1)$. *Assume further that* $\rho < 1$ *and* $f \in \mathcal{A}$. *Let* β *satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\left(\frac{1-g(t)}{2}\right)\mathrm{d}t,$$

where g is the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}t^{1/\nu}(1+g(t)) = \begin{cases} \dfrac{2}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2}\mathrm{d}s, & \gamma > 0, \\ \dfrac{2}{\alpha}\dfrac{t^{1/\alpha-1}}{(1+t)^2}, & \gamma = 0,\ \alpha > 0, \end{cases} \tag{32}$$

with $g(0) = 1$. *Then*

$$\begin{cases} \mathrm{Re}\displaystyle\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)\mathrm{d}t > 0, & \gamma > 0, \\ \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)\mathrm{d}t > 0, & \gamma = 0, \end{cases}$$

if and only if $\mathcal{V}_\lambda(f)$ *is in* $\mathcal{S}^*$ *for* $f \in \mathcal{W}_\beta(\alpha,\gamma)$. *The conclusion does not hold for smaller values of* β.

THEOREM 6.2 *Let* $\Pi_{\mu,\nu}$ *and* Λ_ν *be given as in Theorem* 3.1. *Assume that both* $\Pi_{\mu,\nu}$ *and* Λ_ν *are integrable on* $[0,1]$, *and positive on* $(0,1)$. *Assume further that* $\rho < 1$ *and* $f \in \mathcal{A}$. *Let* $\beta < 1$ *satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)(1-q(t))\mathrm{d}t, \tag{33}$$

where q *is the solution of the initial-value problem* (6). *Then*

$$\begin{cases} \mathrm{Re}\displaystyle\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)\mathrm{d}t > 0, & \gamma > 0, \\ \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)\mathrm{d}t > 0, & \gamma = 0, \end{cases}$$

if and only if $\mathcal{V}_\lambda(f)$ *is in* $\mathcal{CV}$ *for* $f \in \mathcal{W}_\beta(\alpha,\gamma)$. *The conclusion does not hold for smaller values of* β.

Remark 1 Theorems 6.1 and 6.2 yield several known results.

(1) Taking $\gamma = 0$ and $\alpha > 0$, Theorem 6.1 leads to Theorem 2.4 obtained by Balasubramanian et al. in [10, p. 6, $(\mu = 0)$] for the case $\alpha \in [1/2, 1]$, and to Theorem 2 obtained by Ali and Singh in [2] and Theorem 3 in [12, p. 916] for the case $\gamma = 0$, $\alpha = 1$.
(2) Taking $\gamma = 0$ and $\alpha > 0$, Theorem 2.5 leads to Theorem 2.5 obtained by Balasubramanian et al. in [10, p. 7, $(\mu = 0)$] for the case $\alpha \in [1/2, 1]$, and it reduces to Theorem 3 obtained by Ali and Singh in [2] in the case $\gamma = 0$, $\alpha = 1$.

For the case $\alpha = 1 + 2\gamma$, Theorem 6.2 reduces to the following result.

COROLLARY 6.3 *Let* $f \in \mathcal{W}_\beta(1+2\gamma,\gamma) = \mathcal{R}_\gamma(\beta)$, $\gamma \geq 0$, *and let* $\beta < 1$ *satisfy* (33). *Assume* Λ_γ *is defined by* (10) *and* Π_γ *is defined by* (18). *Then*

$$\mathrm{Re}\int_0^1 \Pi_\gamma(t)\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)\mathrm{d}t > 0,$$

where h *is given by* (12), *if and only if* $F(z) = \mathcal{V}_\lambda(f)(z)$ *is in* $\mathcal{CV}$. *The conclusion does not hold for smaller values of* β.

Choosing $\lambda(t) = (1+c)t^c$, Theorem 6.2 leads to the following corollary.

COROLLARY 6.4 *Let* $c > -1$, $\gamma > 0$, $\rho < 1$ *and* $\beta < 1$ *satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = (1+c)\int_0^1 t^c(1-q(t))\mathrm{d}t,$$

where q *is given by* (6). *If* $f \in \mathcal{W}_\beta(\alpha,\gamma)$, *then the function*

$$\mathcal{V}_\lambda(f)(z) = \rho z + (1-\rho)(1+c)\int_0^1 t^{c-1}f(tz)\mathrm{d}t$$

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belongs to $\mathcal{CV}$ provided

$$c \le 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \ge \mu \ge 1.$$

The value of β is sharp.

Additionally, using the sufficient condition (4.4) in Ali et al. [9, p. 816] leads to the following result.

COROLLARY 6.5 *Let $c > -1$, $\rho < 1$ and $\beta < 1$ satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = (1+c)\int_0^1 t^c \left(\frac{1-g(t)}{2}\right) \mathrm{d}t,$$

where g is given by (32). *If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$\mathcal{V}_\lambda(f)(z) = \rho z + (1-\rho)(1+c)\int_0^1 t^{c-1} f(tz)\mathrm{d}t$$

belongs to $\mathcal{S}^*$ provided

$$c \le \begin{cases} 1 + \dfrac{1}{\mu}, & \mu \ge 1 \ (\gamma > 0), \\ 3 - \dfrac{1}{\alpha}, & \gamma = 0, \ \alpha \in (0, 1/3] \cap [1, \infty). \end{cases}$$

The value of β is sharp.

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